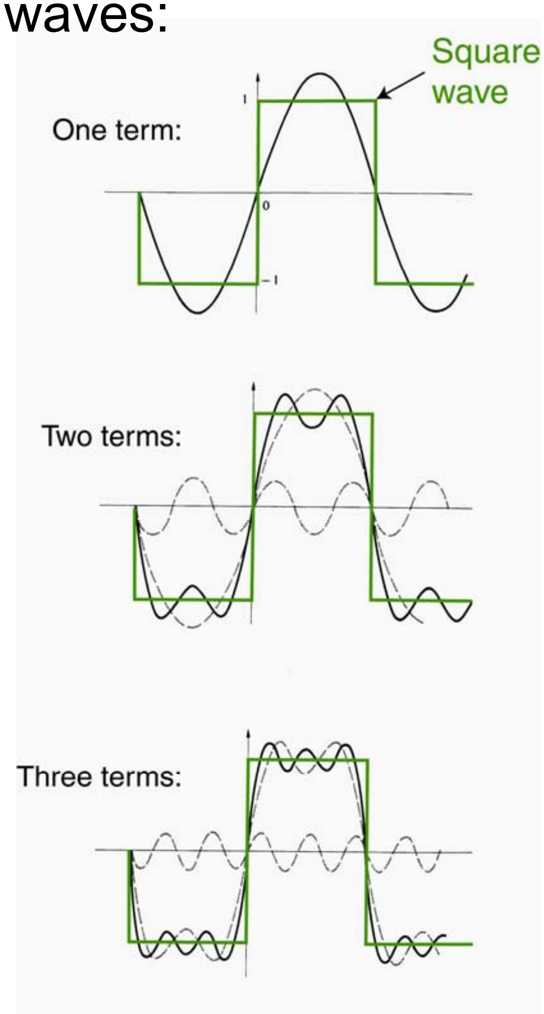
Fourier Theory

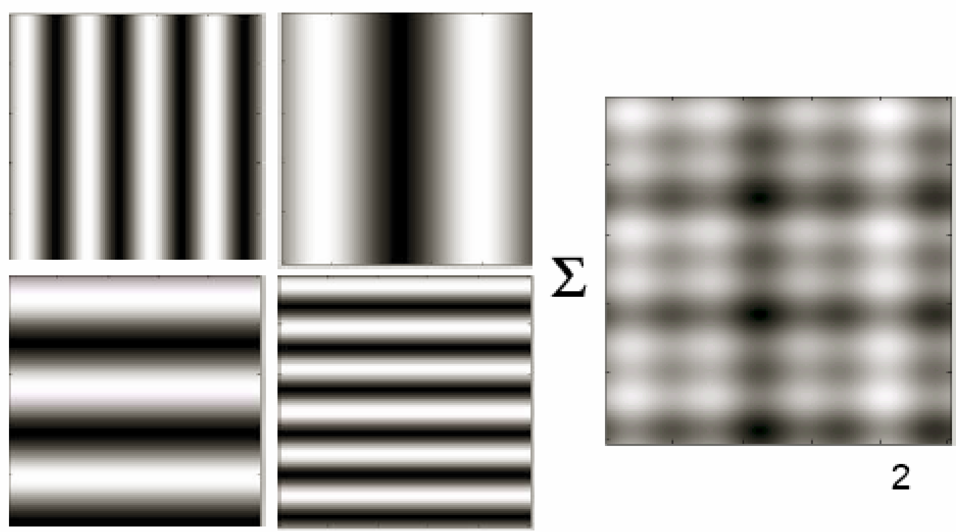
Willy Wriggers
School of Health Information Sciences &
Institute of Molecular Medicine
University of Texas – Houston

Frequency Analysis

Here, we write a square wave as a sum of sine waves:



- Fourier Domain
- Signals (1D, 2D, …) decomposed into sum of signals with different frequencies



Transforms with Functions

Just as we transformed vectors, we can also transform functions:

	Basis Vectors $\{\bar{e}_k[j]\}$	Basis Functions $\{e_k(t)\}$
Transform	$a_k = \bar{v} \cdot \bar{e}_k = \sum_j \bar{v}[j] \cdot \bar{e}_k[j]$	$a_k = f \cdot e_k = \int_{-\infty}^{\infty} f(t) e_k^*(t) dt$
Inverse	$\bar{v} = \sum_k a_k \bar{e}_k$	$f(t) = \sum_k a_k e_k(t)$

The Fourier Transform

Most tasks need an infinite number of basis functions (frequencies), each with their own weight $F(s)$: Harmonics $\{e^{i2\pi st}\}$

	Fourier Series	Fourier Transform
Transform	$a_k = f \cdot e^{i2\pi s_k t} = \int_{-\infty}^{\infty} f(t) e^{-i2\pi s_k t} dt$	$F(s) = f \cdot e^{i2\pi st} = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st} dt$
Inverse	$f(t) = \sum_k a_k e^{i2\pi s_k t}$	$f(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st} ds$

The Fourier Transform

To get the weights (amount of each frequency): $\mathcal{F}$

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st} dt$$

$F(s)$ is the Fourier Transform of $f(t)$: $\mathcal{F}(f(t)) = F(s)$

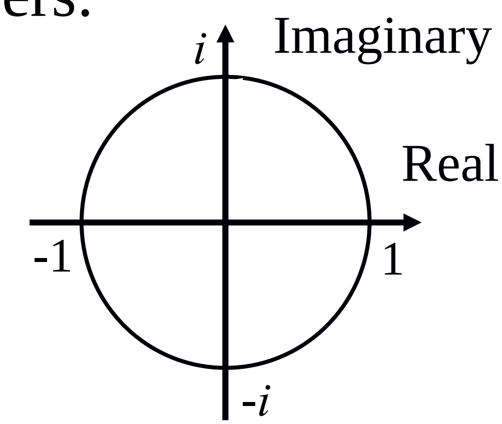
To convert weights back into a signal (invert the transform):

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st} ds$$

$f(t)$ is the Inverse Fourier Transform of $F(s)$: $\mathcal{F}^{-1}(F(s)) = f(t)$

How to Interpret the Weights F(s)

The weights $F(s)$ are complex numbers:

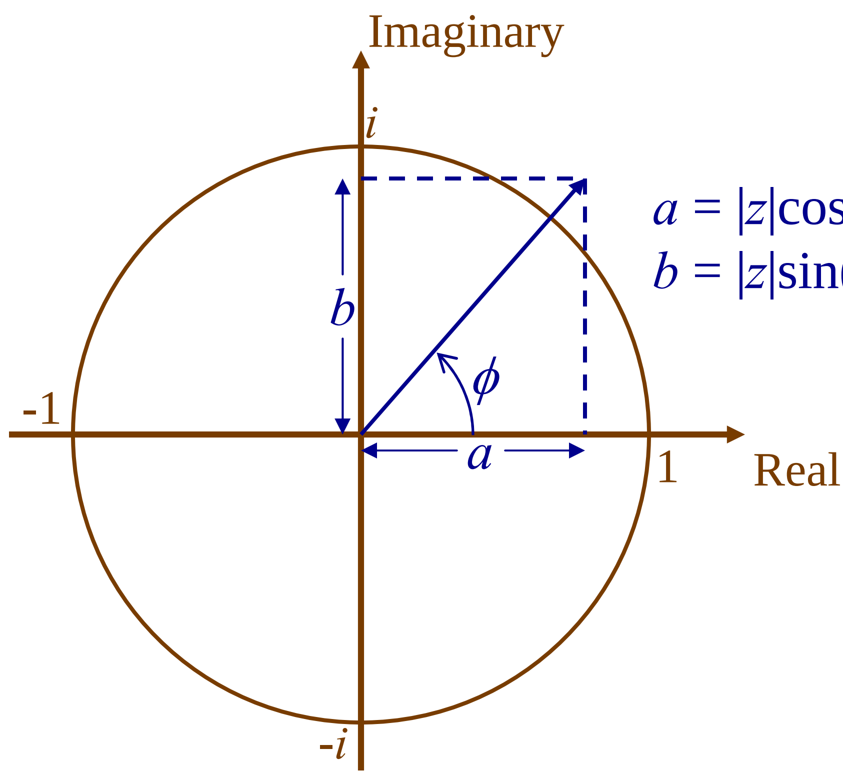


Real part	How much of a *cosine* of frequency *s* you need
Imaginary part	How much of a *sine* of frequency *s* you need
Magnitude	How *much* of a sinusoid of frequency *s* you need
Phase	What *phase* that sinusoid needs to be

Euler's Formula

- Any complex number can be represented using Euler's formula:

$$z = |z|e^{i\phi(z)} = |z|\cos(\phi) + |z|\sin(\phi)i = a + bi$$



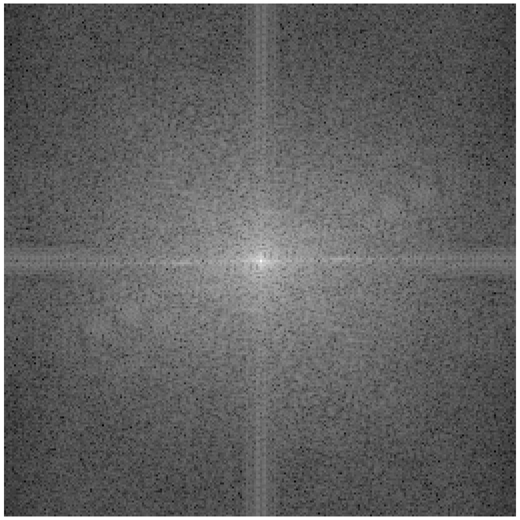
Magnitude and Phase

Remember: complex numbers can be thought of in two ways: (*real*, *imaginary*) or (*magnitude*, *phase*)

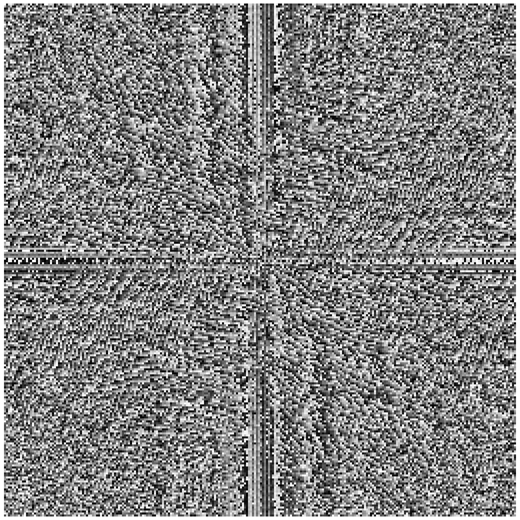
Magnitude: $$|F| = \sqrt{\Re(F)^2 + \Im(F)^2}$$

Phase: $$\phi(F) = \arctan\left(\frac{\Re(F)}{\Im(F)}\right)$$

image



$|F|$



$\phi\,(F)$

Periodic Signals on a Grid

- Periodic signals with period N:
  - Underlying frequencies must also repeat over the period N
  - Each component frequency must be a multiple of the frequency of the periodic signal itself:

$$\frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \cdots$$

- If the signal is discrete:
  - Highest frequency is one unit: period repeats after a single sample
  - No more than N components

$$\frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \cdots \frac{N}{N}$$

Discrete Fourier Transform (DFT)

If we treat a discrete signal with *N* samples as one period of an infinite periodic signal, then

$$F[s] = \frac{1}{N}\sum_{t=0}^{N-1} f[t]e^{-i2\pi st/N}$$

and

$$f[t] = \sum_{s=0}^{N-1} F[s]e^{i2\pi st/N}$$

Note: For a periodic function, the discrete Fourier transform is the same as the continuous transform

- We give up nothing in going from a continuous to a discrete transform as long as the function is periodic
- Computational complexity: $O(N^2)$

Fast Fourier Transform

developed by Tukey and Cooley in 1965

If we let

$$W_N = e^{-i2\pi/N}$$

the Discrete Fourier Transform can be written

$$F[s] = \frac{1}{N}\sum_{t=0}^{N-1} f[t]\cdot W_N^{st}$$

If N is a multiple of 2, $N = 2M$ for some positive integer M, substituting $2M$ for N gives

$$F[s] = \frac{1}{2M}\sum_{t=0}^{2M-1} f[t]\cdot W_{2M}^{st}$$

Fast Fourier Transform

Separating even and odd terms:

$$F[s]=\frac{1}{2}\left\{\frac{1}{M}\sum_{t=0}^{M-1} f[2t]\cdot W_M^{st}+\frac{1}{M}\sum_{t=0}^{M-1} f[2t+1]\cdot W_M^{st}W_{2M}^{s}\right\}$$

Can be written as

$$F[s]=\frac{1}{2}\left\{F_{even}(s)+F_{odd}(s)W_{2M}^{s}\right\}$$

We can use this for the first M terms of the Fourier transform of $2M$ items, then we can re-use these values to compute the last M terms as follows:

$$F[s+M]=\frac{1}{2}\left\{F_{even}(s)-F_{odd}(s)W_{2M}^{s}\right\}$$

Fast Fourier Transform

If M is itself a multiple of 2, do it again!

If N is a power of 2, recursively subdivide until you have one element, which is its own Fourier Transform

```
ComplexSignal FFT(ComplexSignal f) {
  if (length(f) == 1) return f;

  M = length(f) / 2;
  W_2M = e^(-I * 2 * Pi / M)   // A complex value.

  even = FFT(EvenTerms(f));
  odd  = FFT(OddTerms(f));

  for (s = 0; s < M; s++) {
    result[s  ] = even[s] + W_2M^s * odd[s];
    result[s+M] = even[s] – W_2M^s * odd[s];
  }
}
```

Fast Fourier Transform

Computational Complexity:

Discrete Fourier Transform	→	$O(N^2)$
Fast Fourier Transform	→	$O(N \log N)$

Remember: The FFT is just a faster algorithm for computing the DFT — it does not produce a different result

Impulses

One way of probing what a system does is to test it on a single input point (a single spike in the signal, a single point of light, etc.)

Mathematically, a perfect single-point input is written as:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

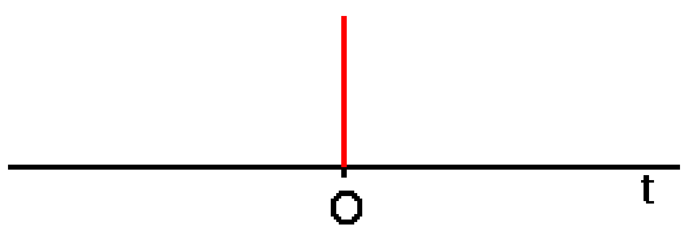
and

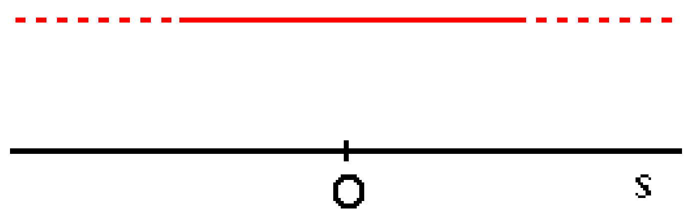
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

This is called the Dirac *delta function*

Delta Function and its FT

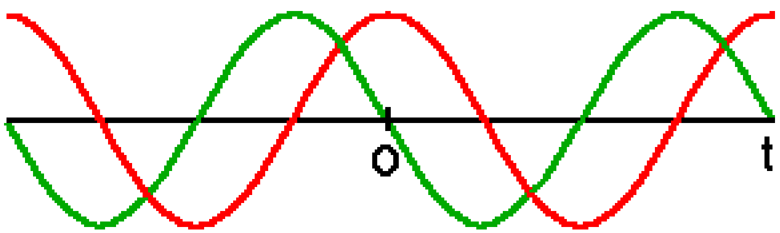
Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\delta(t)$	1

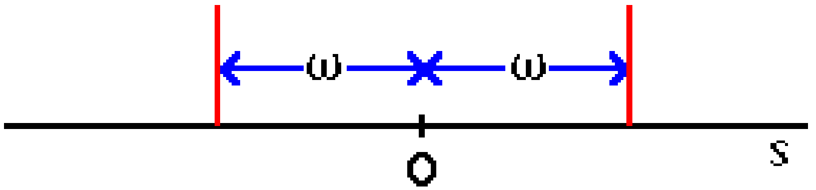




Sinusoids

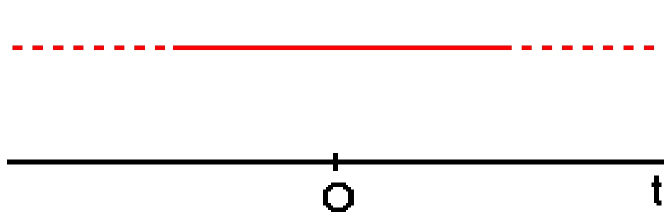
Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\cos(2\pi \omega t)$	$\frac{1}{2}[\delta(s + \omega) + \delta(s - \omega)]$
$\sin(2\pi \omega t)$	$\frac{1}{2}[\delta(s + \omega) - \delta(s - \omega)]i$

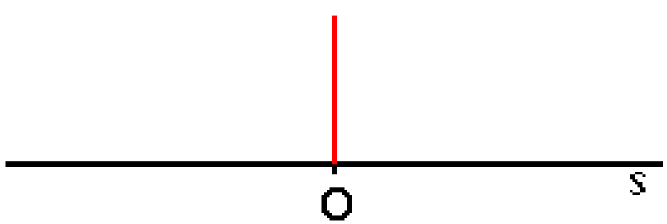




Constant Functions

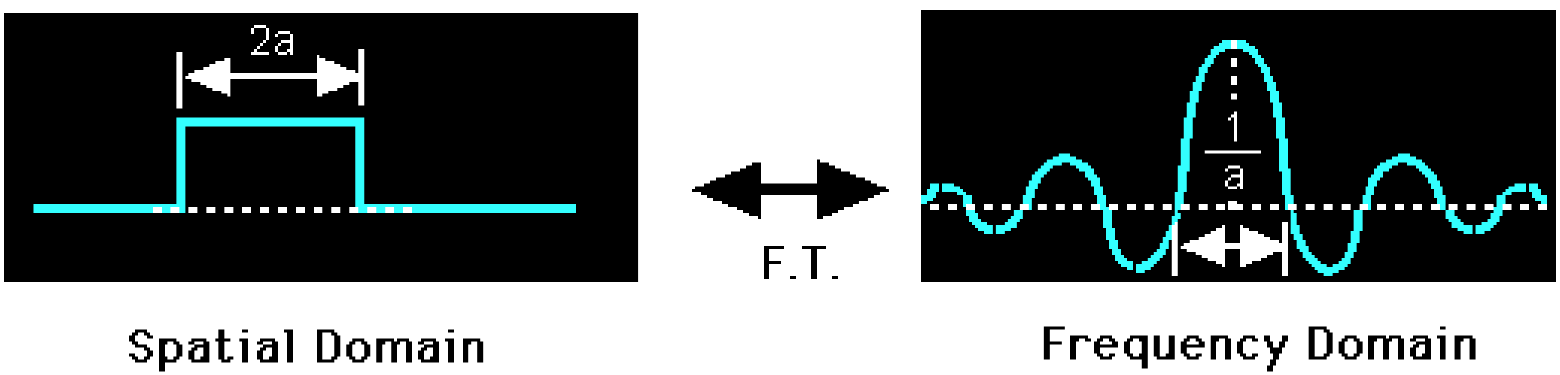
Spatial Domain $f(t)$	Frequency Domain $F(s)$
1	$\delta(s)$
a	$a\,\delta(s)$





Square Pulse

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\Pi_a(t)$	$2a\ \mathrm{sinc}(2as) = \dfrac{\sin(2\pi as)}{\pi s}$

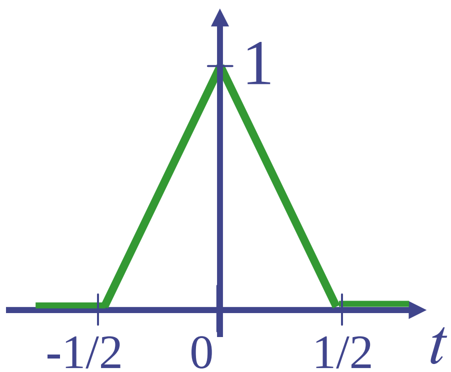


Adapted from http://www.med.harvard.edu/JPNM/physics/didactics/improc/intro/fourier3.html

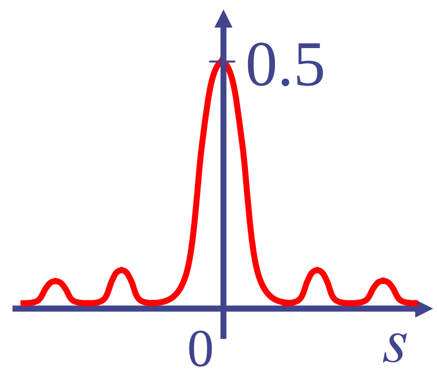
Triangle

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\Lambda_a(t)$	$a\ \mathrm{sinc}^2(as)$

$\Delta_{1/2}(t)$

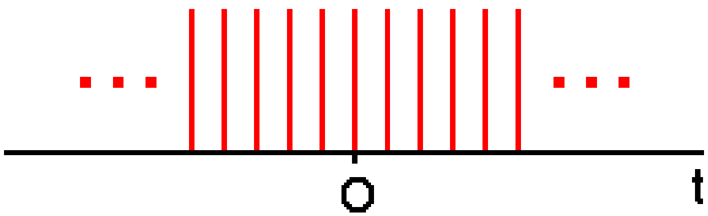


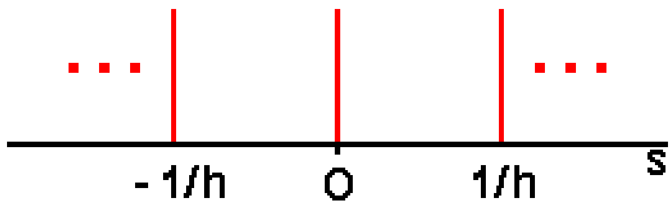
$1/2\ \mathrm{sinc}^2(s/2)$



Comb (Shah) Function

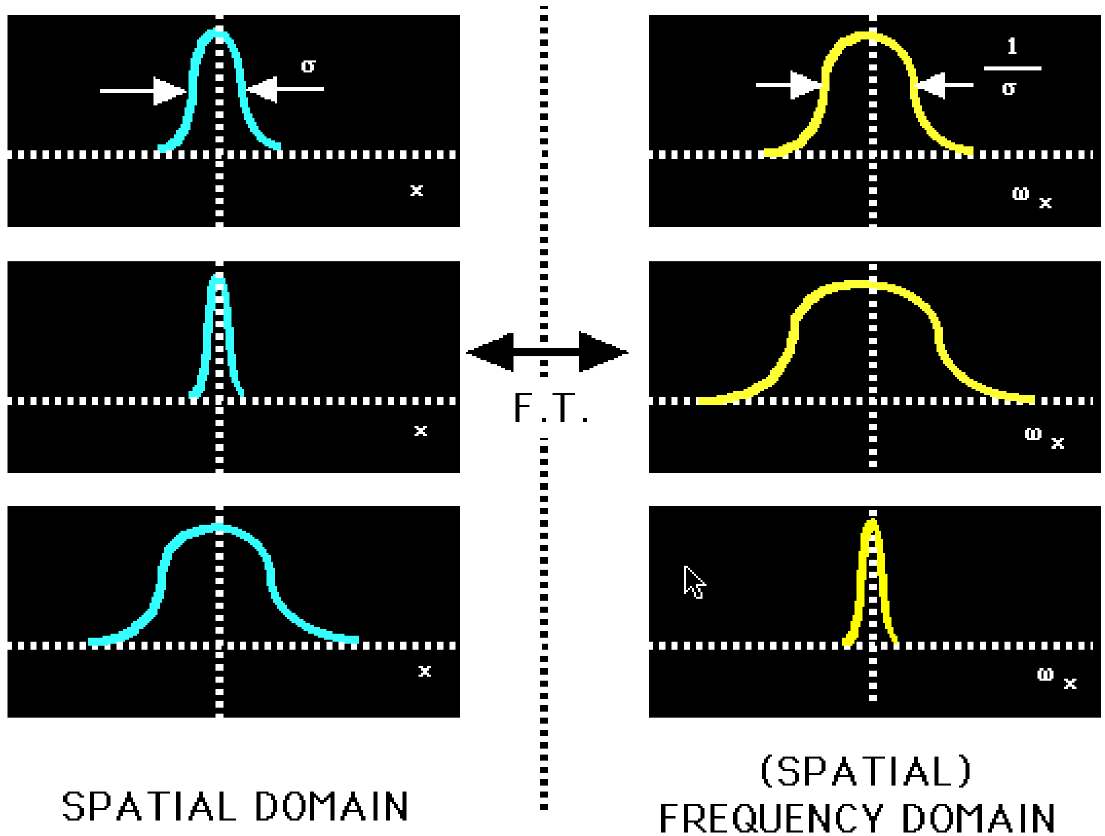
Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\mathrm{comb}_h(t) = \delta(t \bmod h)$	$\delta(t \bmod 1/h)$





Gaussian

$$e^{-\pi\left(\frac{t}{\sigma}\right)^2} \qquad e^{-\pi(\sigma s)^2}$$



http://www.med.harvard.edu/JPNM/physics/didactics/improc/intro/fourier3.html

2D and 3D FTs

The 2D Fourier Transform is linearly separable: the Fourier Transform of a 2D image is the 1D Fourier Transform of the rows followed by the 1D Fourier Transforms of the resulting columns:

$$F[u,v] = \frac{1}{NM}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f[x,y]e^{-i2\pi(ux/N+vy/M)}$$

$$= \frac{1}{NM}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f[x,y]e^{-i2\pi ux/N}e^{-i2\pi vy/M}$$

$$\frac{1}{M}\sum_{y=0}^{M-1}\left[\frac{1}{N}\sum_{x=0}^{N-1} f[x,y]e^{-i2\pi ux/N}\right]e^{-i2\pi vy/M}$$

Similar for 3D!

FT Properties: Addition Theorem

Adding two functions together adds their Fourier Transforms:

$$\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)$$

Multiplying a function by a scalar constant multiplies its Fourier Transform by the same constant:

$$\mathcal{F}(a\, f) = a\, \mathcal{F}(f)$$

Consequence: Fourier Transform is a linear transformation!

FT Properties: Shift Theorem

Translating (shifting) a function leaves the magnitude unchanged and adds a constant to the phase

If $$f_2(t) = f_1(t - a)$$

$$F_1 = \mathcal{F}(f_1)$$

$$F_2 = \mathcal{F}(f_2)$$

then

$$|F_2| = |F_1|$$

$$\phi(F_2) = \phi(F_1) - 2\pi s a$$

Intuition: magnitude tells you "how much",
phase tells you "where"

FT Properties: Scaling Theorem

Scaling a function's abscissa (domain or horizontal axis) inversely scales the both magnitude and abscissa of the Fourier transform.

If
$$f_2(t) = f_1(a\,t)$$
$$F_1 = \mathcal{F}(f_1)$$
$$F_2 = \mathcal{F}(f_2)$$

then
$$F_2(s) = (1/|a|)\,F_1(s\,/\,a)$$

FT Properties: Rotation

Rotating a 2-D function rotates it's Fourier Transform

If

$$f_2 = \text{rotate}_\theta(f_1)$$
$$= f_1(x\cos(\theta) - y\sin(\theta),\, x\sin(\theta) + y\cos(\theta))$$

$$F_1 = \mathcal{F}(f_1)$$

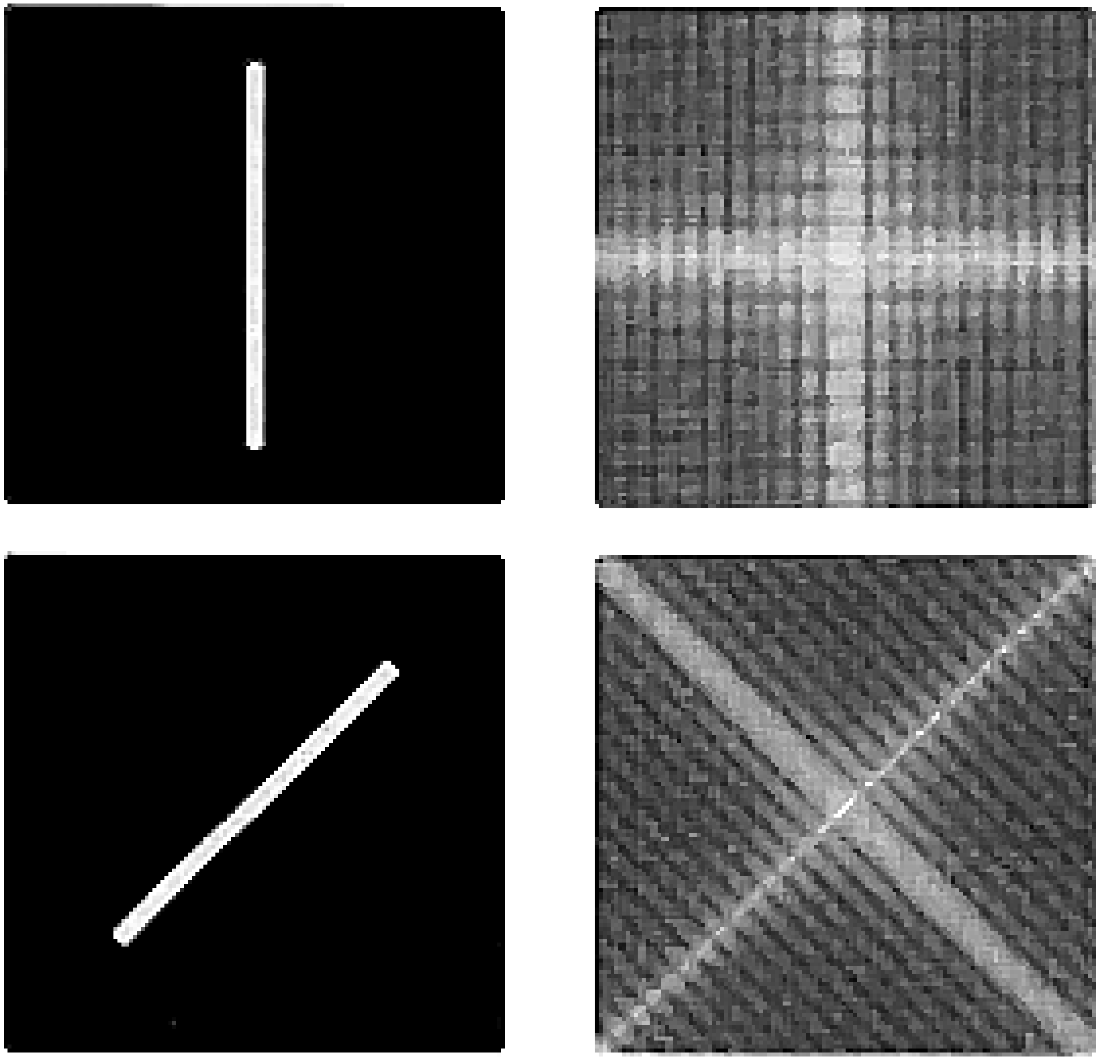
$$F_2 = \mathcal{F}(f_2)$$

then

$$F_2(s) = F_1(x\cos(\theta) - y\sin(\theta),\, x\sin(\theta) + y\cos(\theta))$$

i.e., the Fourier Transform is rotationally invariant.

Rotation Invariance (sort of)



needs
more
boundary
padding!

© http://mail.udlap.mx/~oldwall/docencia/IMAGENES/chapter2/image_232_IS548.html

Convolution

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

FT Properties: Convolution Theorem

Convolution

$$f(t) * g(t) \leftrightarrow F(s)\ G(s)$$

Correlation

$$f(t) * g(-t) \leftrightarrow F(s)\ G^{*}(s)$$

FT Properties: Convolution Theorem

Let F, G, and H denote the Fourier Transforms of signals f, g, and h respectively

$g = f(t) * h(t)$ implies $G = F(s)\, H(s)$

$g = f(t)\, h(t)$ implies $G = F(s) * H(s)$

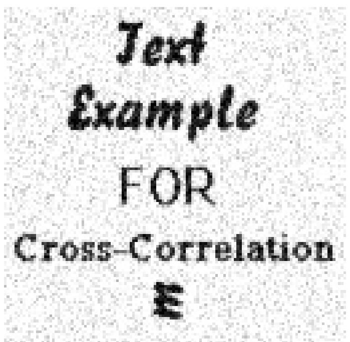
$g = f(t) * h(-t)$ implies $G = F(s)\, H^{*}(s)$

$g = f(t)\, h(-t)$ implies $G = F(s) * H^{*}(s)$

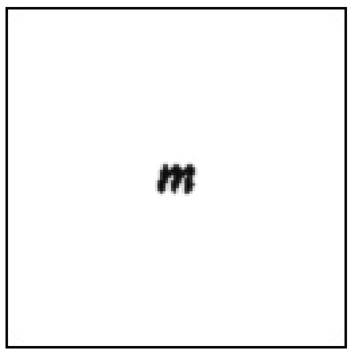
Convolution in one domain is multiplication in the other and vice versa

Template “Convolution”

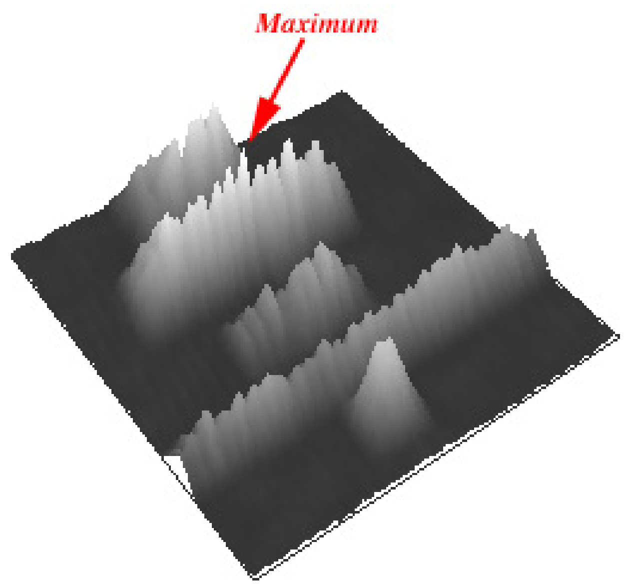
- Actually, is a correlation method
- Goal: maximize correlation between target and probe image
- Here: only translations allowed but rotations also possible



target



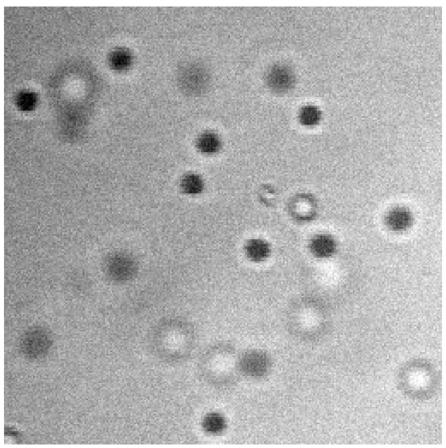
probe



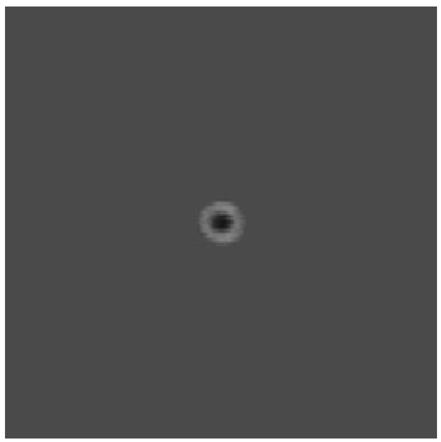
Particle Picking

•Use spherical, or rotationally averaged probes
•Goal: maximize correlation between target and probe image

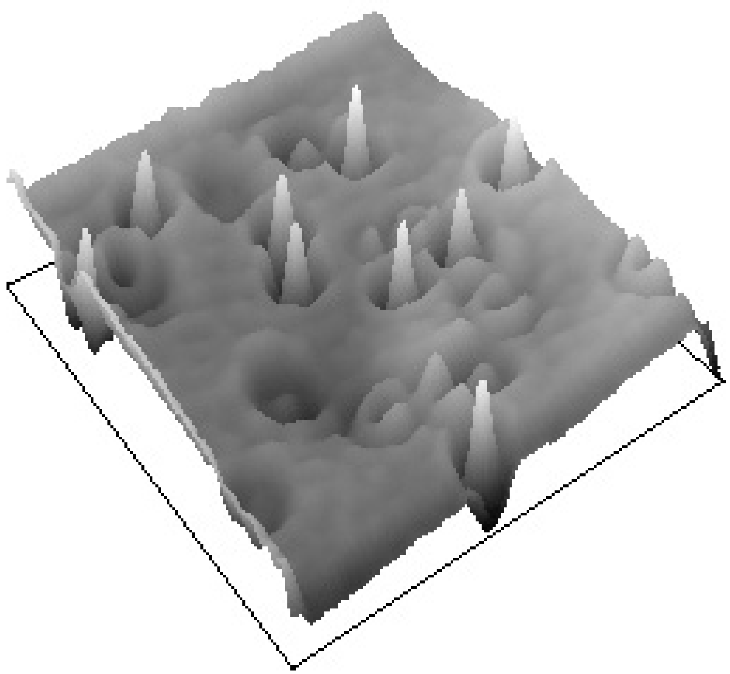
microscope image of latex spheres



target



probe



Power Spectrum

The power spectrum of a signal is the Fourier Transform of its *autocorrelation function*:

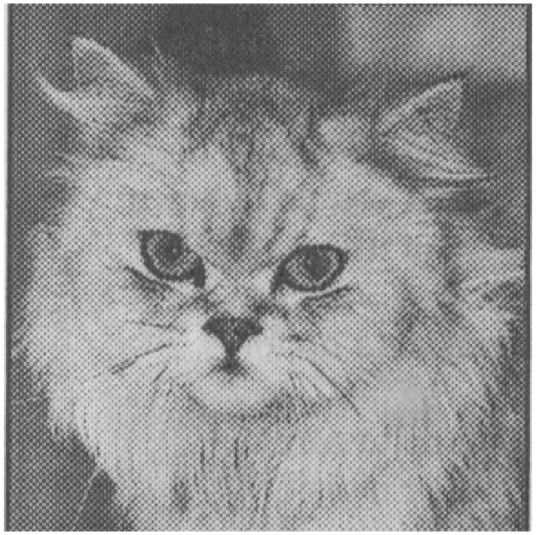
$$
\begin{aligned}
P(s) &= \mathcal{F}(f(t) * f(-t)) \\
&= F(s)\, F^{*}(s) \\
&= |F(s)|^{2}
\end{aligned}
$$

It is also the squared magnitude of the Fourier transform of the function

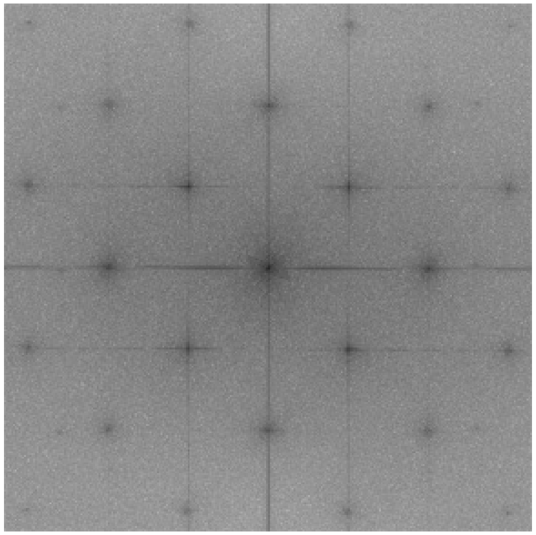
It is entirely real (no imaginary part).

Useful for detecting periodic patterns / texture in the image.

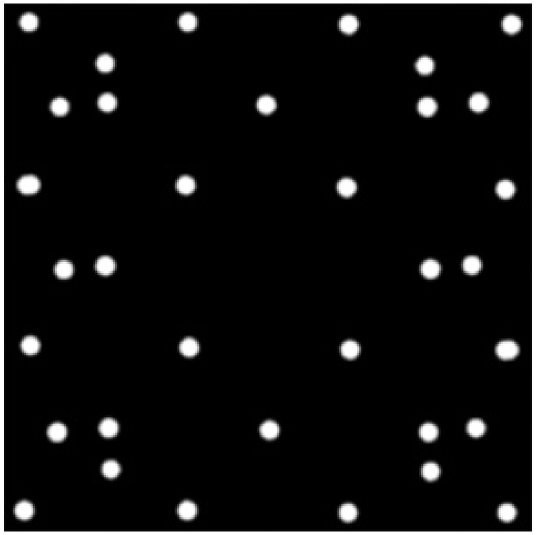
Use of Power Spectrum in Filtering



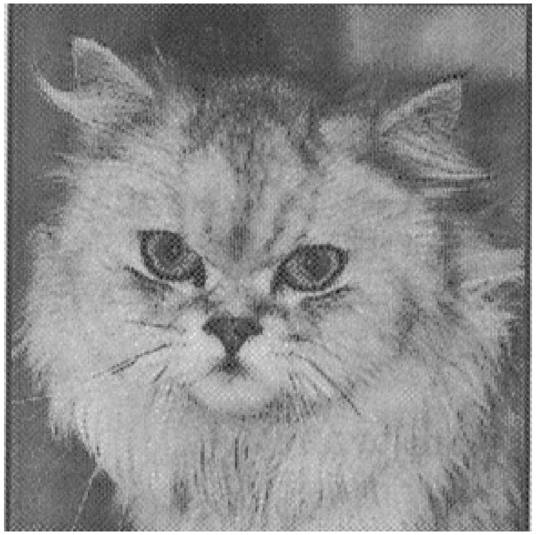
Original with noise patterns



Power spectrum showing noise spikes



Mask to remove periodic noise



Inverse FT with periodic noise removed

FT Properties: Rayleigh's Theorem

Total sum of squares is the same in either domain:

$$\int_{-\infty}^{\infty} \left| f(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| F(s) \right|^2 ds = \int_{-\infty}^{\infty} P(s)\, ds$$

Figure and Text Credits

Text and figures for this lecture were adapted in part from the following source, in agreement with the listed copyright statements:

http://web.engr.oregonstate.edu/~enm/cs519

Resources

Textbooks:
Kenneth R. Castleman, Digital Image Processing, Chapter 10
John C. Russ, The Image Processing Handbook, Chapter 5